On the Laplace transform of perpetuities with thin tails

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Unpublished note – December 23, 2009

Abstract

We consider the random variables \( R \) which are solutions of the distributional equation \( R \overset{\text{d}}{=} MR + Q \), where \((Q, M)\) is independent of \( R \) and \(|M| \leq 1\). Goldie and Gröbel showed that the tails of \( R \) are no heavier than exponential. Alsmeyer and al provide a complete description of the domain of the Laplace transform of \( R \). We present here a simple proof in a particular case and an extension to the Markovian case.

AMS Classification 2000: Primary 60H25; secondary 60E99

1 Introduction

We define on some probability space \((\Omega, \mathcal{A}, \mathbb{P})\) a couple of random variables \((M, Q)\), a sequence \((M_n, Q_n)_{n \geq 0}\) of independent and identically distributed random vectors with the same law as \((M, Q)\), and \(R_0\) a random variable independent of the sequence \((M_n, Q_n)_{n \geq 0}\). Define the sequence \((R_n)_{n \geq 0}\) by

\[
R_{n+1} = M_nR_n + Q_n,
\]

for any \(n \geq 0\). This sequence has been extensively studied in the last decades. Under weak assumptions (see [8]) which are obviously fulfilled in our setting, it can be shown that the sequence \((R_n)_{n \geq 0}\) converges almost surely to a random variable \(R\) such that

\[
R \overset{\text{d}}{=} MR + Q,
\]

where \(R\) is independent of \((M, Q)\).

In [7], Kesten established that \(R\) is in general heavy-tailed (i.e. not all the moments of \(R\) are finite) even if \(Q\) is light-tailed as soon as \(|M|\) can be greater than 1. Nevertheless, Goldie and Gröbel [4] have shown that \(R\) can have some exponential moments if \(|M| \leq 1\). In particular, if \(Q\) and \(M\) are nonnegative the following result holds.

**Theorem 1.1** (Goldie, Gröbel [4]). Assume that

\[
\mathbb{P}(Q \geq 0, 0 \leq M \leq 1) = 1, \quad \mathbb{P}(M < 1) > 0
\]

and that there is \(v_Q > 0\) (possibly infinite) such that

\[
\mathbb{E}(e^{vQ}) \begin{cases} < +\infty & \text{if } v < v_Q, \\ = +\infty & \text{if } v > v_Q. \end{cases}
\]

Then, the Laplace transform \(v \mapsto \mathbb{E}(e^{vR})\) of the solution \(R\) of \((2)\) is finite on the set \((-\infty, v_{GG})\) with \(v_{GG} = v_Q \wedge \sup\{v \geq 0, \mathbb{E}(e^{vQ}M) < 1\}\).
In fact, the domain of the Laplace transform of $R$ is larger than $(-\infty, v_{GG})$. In [1], a full description of this domain is established. Let us provide a simple proof under the assumptions of Theorem 1.1.

## 2 The main result

**Theorem 2.1.** Under the assumptions of Theorem 1.1 assuming furthermore that $R_0$ is non-negative and has all its exponential moments finite, then

$$\sup_{n \geq 0} \mathbb{E}(e^{vR_n}) < +\infty \quad \text{and} \quad \mathbb{E}(e^{vR}) < +\infty$$

for any $v < v_c$, where

$$v_c = v_Q \wedge \sup\{v \geq 0, \mathbb{E}(e^{vQ1_{\{M=1\}}}) < 1\}.$$  

Moreover, for any $v > v_c$, $\sup_{n \geq 0} \mathbb{E}(e^{vR_n}) = +\infty$ and $\mathbb{E}(e^{vR}) = +\infty$.

For other recent generalizations of [4], the interested reader is referred to [6], where the authors give sharper results than ours on the tails for some specific examples.

**Proof of Theorem 2.1.** Let us start this section with the main lines of the proof of Theorem 1.1 of Goldie and Gr"ubel [4]. For $\rho > 0$, let $M_\rho$ be the set of probability measures on $\mathbb{R}_+$ with finite exponential moment of order $\rho$, and $d_\rho$ a distance defined on $M_\rho$ by: for $\mu, \nu \in M_\rho$,

$$d_\rho(\mu, \nu) = \int_0^\infty e^{\rho u} |\mu[0, u) - \nu[0, u)| \, du.$$  

Define the application $T$ on $M_\rho$ as follows: for $X$ with law $\mu \in M_\rho$, $T \mu$ is the law of $Q + M_X$ with $(M, Q)$ independent of $X$. It is shown in [4] that,

$$d_\rho(T \mu, T \nu) \leq \mathbb{E}(e^{\rho Q}M)d_\rho(\mu, \nu).$$

Since

$$\mathbb{E}(e^{vQ}) = v \int_0^\infty e^{vu} \mathbb{P}(X \geq u) \, du,$$

one can show that, for any $n \geq 0$ and $v < \min(v_0, v_Q)$ with $v_0 = \sup\{v \geq 0, \mathbb{E}(e^{vQ}M) < 1\}$,

$$\mathbb{E}(e^{vR_n}) \leq v \frac{1 - \mathbb{E}(e^{vQ}M)^n}{1 - \mathbb{E}(e^{vQ}M)} d_v(T \mu_0, \mu_0) + \mathbb{E}(e^{vR_0}).$$

In other words, Goldie and Gr"ubel [4] established that for any $v < \min(v_0, v_Q)$, $(\mathbb{E}(e^{vR_n}))_n$ is uniformly bounded. This estimate can be extended to a larger domain.

Let us define $v_1 = \sup\{v \geq 0, \mathbb{E}(e^{vQ}1_{\{M=1\}}) < 1\}$ and $v_c = \min(v_1, v_Q)$. Let us fix $v < v_c$ and choose $\varepsilon > 0$ such that

$$\rho := \mathbb{E}(e^{vQ}1_{\{1-\varepsilon < M \leq 1\}}) < 1.$$
Then we get, for any \( n \geq 0 \),
\[
L_{n+1}(v) := E(e^{v R_{n+1}}) = E(e^{v(M_n R_n + Q_n)})
\leq E(e^{v((1-\varepsilon) R_n + Q_n) 1\{M_n \leq 1-\varepsilon\}}) + E(e^{v(R_n + Q_n) 1\{1-\varepsilon < M_n \leq 1\}})
\leq L_n((1-\varepsilon)v) L_Q(v) + \rho L_n(v)
\]
where \( L_Q(v) = E(e^{v Q}) \). By iteration of this estimate, one gets for any \( n \geq 0 \)
\[
L_n(v) \leq \left( \sum_{k=0}^{n-1} \rho^k L_{n-k}((1-\varepsilon)v) \right) L_Q(v) + \rho^n L_0(v) .
\]
Let us notice that we have in fact more: for the same \( \varepsilon \), and for any \( \tilde{v} \leq v \), \( \tilde{\rho} := E(e^{\varepsilon Q 1\{1-\varepsilon < M \leq 1\}}) < \rho \), hence, by the same method as before,
\[
L_n(\tilde{v}) \leq \left( \sum_{k=0}^{n-1} \rho^k L_{n-k}((1-\varepsilon)\tilde{v}) \right) L_Q(\tilde{v}) + \rho^n L_0(\tilde{v}) . \tag{4}
\]
Let us define \( \bar{L} = \sup_{n \geq 0} L_n \). Taking the supremum over \( n \) in (4), one gets for any \( \tilde{v} \leq v \)
\[
\bar{L}(\tilde{v}) \leq \frac{1}{1 - \rho} \bar{L}((1-\varepsilon)\tilde{v}) L_Q(\tilde{v}) + L_0(\tilde{v}) . \tag{5}
\]
There is \( k \in \mathbb{N} \) such that \( (1-\varepsilon)^k v < v_0 \), hence \( \bar{L}((1-\varepsilon)^k v) < +\infty \). Applying \( k \) times estimate (4), one then obtains immediatly that \( \bar{L}(v) < +\infty \), which achieves the first part of the proof.

On the other hand, if \( v > v_Q \), \( R_1 \geq Q_0 \) immediatly implies \( L_1(v) = +\infty \); if \( v > v_1 \), \( \rho_0 := E(e^{v Q 1\{M = 1\}}) > 1 \) (except in a trivial case, left to the reader) and, for all \( n \geq 0 \),
\[
L_{n+1}(v) \geq \rho_0 L_n(v) ,
\]
implying that \( \bar{L}(v) = +\infty \). \( \square \)

3 Some extensions and perspectives

What happens if the random variables \((M_n, Q_n)_{n \geq 0}\) are no longer independent? We provide here a partial result under a Markovian assumption when the contractive term \( M \) is less than 1.

Let us introduce \( X = (X_n)_{n \geq 0} \) an irreducible recurrent Markov process with finite space \( E \) and \((\langle M_n(x), Q_n(x) \rangle_{x \in E})_{n \geq 0}\) a sequence of i.i.d. random vectors supposed to be independent of \( X \). We assume that, for all \( x \in E \),
\[
\mathbb{P}(0 \leq M(x) < 1) = 1 ,
\]
but we do not assume in the sequel that \( Q \) is non negative. The sequence \((R_n)_{n \geq 0}\) is defined by
\[
R_{n+1} = M_n(X_n) R_n + Q_n(X_n) ,
\]
\( R_0 \) being arbitrary (with all exponential moments). Notice that the process \((X_n, R_n)_{n \geq 0}\) is a Markov process whereas \((R_n)_{n \geq 0}\) is not (in general).
Proposition 3.1. Introduce \( \underline{v} = \inf_{x \in E} v|Q(x)| \), with \( v|Q(x)| \) defined as in (5). For any \( \underline{v} < v \),
\[
\sup_{n \geq 0} E\left( e^{v|R_n|}\right) < +\infty.
\]
Moreover, if \( v > \underline{v} \), then this supremum is infinite.

Proof. Let us introduce \( \overline{M}_n = \max_{x \in E} M_n(x) \) and \( \overline{Q}_n = \max_{x \in E} |Q_n(x)| \). The random variables \((\overline{M}_n, \overline{Q}_n)\) are i.i.d. Define the sequence \((\overline{R}_n)_{n \geq 0}\) by
\[
\overline{R}_0 = |R_0| \quad \text{and} \quad \overline{R}_{n+1} = \overline{M}_n \overline{R}_n + \overline{Q}_n \quad \text{for} \quad n \geq 1.
\]
Obviously, \( |R_n| \leq \overline{R}_n \) for all \( n \geq 0 \). Thus it is sufficient to study the Laplace transforms of \((\overline{R}_n)_{n \geq 0}\). On the other hand, Theorem 2.1 ensures that \( (E\left( e^{v|\overline{R}_n|}\right) ) \) is uniformly bounded as soon as \( v < \overline{v}_c = \min(\overline{v}_1, v_{\max}) \) with \( \overline{v}_1 = \sup\{v \geq 0 : E\left( e^{v|Q_1|/|M_1|}\right) < 1\} > 0 \). In our case, \( \overline{v}_1 \) is infinite since \( P(\overline{M} < 1) = 1 \). At last, for \( v \geq 0 \),
\[
\sup_{x \in E} E\left( e^{v|Q(x)|}\right) \leq E\left( e^{vQ}\right) = E\left( \sup_{x \in E} e^{v|Q(x)|}\right) \leq \sum_{x \in E} E\left( e^{v|Q(x)|}\right).
\]
Thus \( v_{\max} = \inf_{x \in E} v|Q(x)| \).

On the other hand, choose \( v > \underline{v} \). There exists \( x_0 \in E \) such that \( E\left( e^{v|Q(x_0)|}\right) \) is infinite. Then, for any \( n \geq 0 \),
\[
E\left( e^{v|R_{n+1}|}\right) \geq E\left( e^{v|R_n|}\right) I_{\{X_n=x_0\}}
\]
\[
\geq E\left( e^{-v|R_n|} e^{v|Q_n(x_0)|}\right) I_{\{X_n=x_0\}}
\]
\[
\geq E\left( I_{\{X_n=x_0\}} e^{-v|R_n|}\right) E\left( e^{v|Q_n(x_0)|}\right).
\]

The recurrence of \( X \) ensures that \( \{n \geq 0, E\left( e^{v|R_n|}\right) = +\infty\} \) is infinite.

Remark 3.2. In [2], we use the previous estimates to improve the results of [3, 5] on the tails of the invariant measure of a diffusion process with Markov switching.

References


Compiled December 23, 2009.

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